

Horizon thermodynamics and spacetime mappings

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When black holes are dynamical, event horizons are replaced by apparent and trapping horizons. Conformal and Kerr-Schild transformations are widely used in relation with dynamical black holes and we study the behaviour under such transformations of quantities related to the thermodynamics of these horizons, such as the Misner-Sharp-Hernandez mass (internal energy), the Kodama vector, surface gravity, and temperature. The transformation properties are not those expected on the basis of naive arguments.

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I. INTRODUCTION

Black holes are one of the most intriguing predictions of General Relativity, which persist in alternative theories of gravity motivated by quantum considerations and by cosmology. Black holes have been shown to play very important roles in the dynamics and evolution of astrophysical systems and to be an important source of gravitational waves, which are expected to be detected experimentally within the next decade. In this regard, the prediction of accurate templates is a necessary condition for recovering a gravitational wave signal from the background noise. Waves generated during black hole collapse and the inspiral and merger of a binary system containing at least one black hole are notoriously difficult to predict accurately and the most sophisticated tools of approximate analytical and numerical relativity are used to attack the problems of the dynamics and wave generation in these systems. In numerical relativity, in particular, the “black hole horizon” is located using marginally trapped surfaces and apparent and trapping horizons (*e.g.*, [1]): the teleological event horizon requires the knowledge of the entire future of the spacetime, which is impossible in a numerical calculation. On the contrary, the theories of black hole dynamics and thermodynamics constructed during the 1960s and 1970s [2–4] rely on the concept of event horizon, which is well suited to static and stationary black holes. When black holes are dynamical, the concept of event horizon defining the black hole itself appears to be useless for practical purposes. It has been suggested that apparent horizons replace event horizons even though apparent horizons have two drawbacks: first, they depend on the spacetime foliation [5, 6] and, sec-

ond, in some of the analytical solutions known to exhibit time-dependent apparent horizons, these can be timelike instead of spacelike.

It is claimed that it is the thermodynamics of apparent, not of event, horizons which are physically meaningful [7, 8]. However, while for stationary black holes several calculational methods give the same results for the surface gravity, the Hawking temperature and other thermodynamical quantities, this is not the case for dynamical black holes and for apparent/trapping horizons. There are several possibilities to define the surface gravity and horizon temperature and it is not clear which alternative is the physical one (see for example the reviews [9, 10]). The issue is not an easy one to settle and the debate will likely remain for some time. At the same time, the thermodynamics of apparent horizons are being criticized (for example, apparent horizons may not satisfy a quantum-generalized second law [11]), but apparent horizons and marginally trapped surfaces are all that is used to locate the black hole “horizon” in numerical relativity [1]. Here we do not argue in favour of one prescription or the other but we focus on providing tools to understand the various proposals for thermodynamical quantities (internal energy [12–15], Kodama vector, Kodama surface gravity [16], and temperature).

It is useful to have explicit examples of dynamical black holes to use as testbeds for alternative definitions and for new approaches. While analytical solutions describing dynamical black holes are rare, a few are known which are interpreted as black holes embedded in cosmological backgrounds, in General Relativity and in alternative theories of gravity (see Ref. [17] for a recent review). These are all spherically symmetric solutions and, because of their relative simplicity, we also focus on spherical symmetry in this paper. Another important reason is that, in General Relativity, the Misner-Sharp-Hernandez mass is defined only in spherical symmetry. An added advantage is that in spherical symmetry one has pre-

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ferred spacetime slicings and the problem of the foliation-dependence of the apparent horizons is alleviated. We discuss two quantities appearing in black hole thermodynamics, the Misner-Sharp-Hernandez mass (which coincides with the Hawking-Hayward quasi-local energy in spherical symmetry and is identified with the internal energy U in the first law $TdS = dU + dW$) and the Kodama vector (which defines a Kodama surface gravity and horizon temperature entering the first law of thermodynamics).

Within General Relativity, and even more in scalar-tensor and $f(R)$ gravity, conformal transformations are a useful mathematical tool to relate analytical solutions. It is not surprising, therefore, that following earlier literature on the interplay between conformal transformations and horizon thermodynamics in asymptotically flat spacetimes in various theories of gravity [18–20] and for conformal Killing horizons [21], recent papers have discussed the thermodynamics of apparent horizons in relation with conformal transformations [22–28]. Another issue is the quantum state of the scalar field used to compute the Hawking effect [29]: in general the vacuum state will be changed by a conformal transformation.

The first and most obvious property of conformal transformations in this context is that the location of apparent horizons changes under conformal transformations or apparent horizons are created where there was none.¹ On the contrary event horizons are null surfaces and are unaffected by conformal rescalings. A precise way to recover apparent horizons under conformal mappings was discussed in Ref. [26]: a surface of zero entropy expansion should be considered instead of a trapping horizon with vanishing area expansion. This procedure preserves the thermodynamical properties of quasi-local horizons under conformal transformations [26]. This non-invariance of the apparent horizons should be kept in mind in the following discussion in Sec. II. For dynamical horizons, the Kodama vector is often used in place of a timelike Killing vector to identify a preferred notion of surface gravity and horizon temperature. The transformation of these quantities under conformal transformations is discussed in Sec. III.

Another technique which is used to generate cosmological black hole solutions starting from stationary black hole metrics as “seeds” is that of the Kerr-Schild transformation. The transformation properties of the relevant thermodynamical quantities under Kerr-Schild transformations is discussed in Sec. IV, while Sec. V contains our conclusions.

¹ For example, the Husain-Martinez-Nuñez scalar field solution of General Relativity [30], which hosts a black hole in a portion of the spacetime manifold, is conformal to the Fisher-Janis-Newman-Winicour-Wyman solution which describes a naked singularity [31].

II. BEHAVIOR OF THE MISNER-SHARP-HERNANDEZ MASS UNDER CONFORMAL TRANSFORMATIONS

The Misner-Sharp-Hernandez mass M_{MSH} [12, 13], which coincides with the Hawking-Hayward quasi-local mass [14, 15] in spherical symmetry, is usually identified with the internal energy U of a black hole system, which enters the first law of thermodynamics $TdS = dU + dW$ in General Relativity with spherical symmetry. Here we derive a general law for the transformation of the Misner-Sharp-Hernandez mass under conformal transformations of the metric $g_{ab} \rightarrow \tilde{g}_{ab} = \Omega^2 g_{ab}$. We consider a general spherically symmetric line element of the form

$$ds^2 = -A(t, R)dt^2 + B(t, R)dR^2 + R^2 d\Omega_{(2)}^2, \quad (1)$$

where R is the areal radius determined by the 2-spheres of symmetry and $d\Omega_{(2)}^2 = d\theta^2 + \sin^2 \theta d\varphi^2$ is the line element on the unit 2-sphere. The conformal transformation is assumed to preserve the spherical symmetry, *i.e.*, $\Omega = \Omega(t, R)$. Note that, in general, if the metric g_{ab} is a solution of the Einstein equations with a “reasonable” form of matter, \tilde{g}_{ab} is not a solution for the same form of matter, or the corresponding form of mass-energy can be completely unrealistic. A vacuum solution g_{ab} will be mapped into a non-vacuum solution. However, the use of conformal transformations is so widespread even in GR to motivate the following analysis.

The line element (1) is mapped into

$$d\tilde{s}^2 = \Omega^2 ds^2 = -\Omega^2 A dt^2 + \Omega^2 B dR^2 + \tilde{R}^2 d\Omega_{(2)}^2, \quad (2)$$

where

$$\tilde{R} = \Omega R \quad (3)$$

is the “new” areal radius. It is well known since the early work of Dicke [32] that, under a conformal transformation, lengths and time intervals scale as Ω (which is consistent with eq. (3)) while masses and energies scale as Ω^{-1} . Therefore, one would naively expect the Misner-Sharp-Hernandez mass to scale as Ω^{-1} and the Hawking temperature to scale the same way since $k_B T$ (where k_B is the Boltzmann constant) is dimensionally an energy. The situation is, however, more complicated: Dicke’s dimensional argument [32] applies to test particles but the Misner-Sharp-Hernandez/Hawking-Hayward mass includes gravitational energy and other energy contributions, and the Hawking temperature results from a process involving a quantum test field, hence classical test particle considerations may not be sufficient (indeed, we will find that this is the case, at least for the mass M_{MSH}).

To find the transformation properties of the Misner-Sharp-Hernandez mass M_{MSH} of a sphere of radius R , consider its definition in the tilded and in the non-tilded worlds,

$$1 - \frac{2\tilde{M}_{MSH}}{\tilde{R}} = \tilde{g}^{ab} \tilde{\nabla}_a \tilde{R} \tilde{\nabla}_b \tilde{R} \quad (4)$$

and

$$1 - \frac{2M_{MSH}}{R} = g^{ab} \nabla_a R \nabla_b R. \quad (5)$$

Using eq. (3), eq. (4) gives

$$\Omega^{-2} g^{ab} \nabla_a (\Omega R) \nabla_b (\Omega R) = 1 - \frac{2\tilde{M}_{MSH}}{\Omega R}, \quad (6)$$

from which it follows that

$$\tilde{M}_{MSH} = \Omega M_{MSH} - \frac{R^3}{2\Omega} \nabla^a \Omega \nabla_a \Omega - R^2 \nabla^a \Omega \nabla_a R. \quad (7)$$

The conformal transformation turns geometry (Ω and its gradient) into quasi-local mass-energy, as is evident from the fact that a vacuum solution is turned into a non-vacuum one due to the transformation property of the Ricci tensor [2, 33, 34]

$$\begin{aligned} \tilde{R}_{ab} = & R_{ab} - 2\nabla_a \nabla_b (\ln \Omega) - g_{ab} g^{ef} \nabla_e \nabla_f (\ln \Omega) \\ & + 2\nabla_a (\ln \Omega) \nabla_b (\ln \Omega) - 2g_{ab} g^{ef} \nabla_e (\ln \Omega) \nabla_f (\ln \Omega) \end{aligned} \quad (8)$$

(the ambiguity in the separation of matter and geometry is a recurrent feature of alternative theories of gravity [35]). Definitely, M_{MSH} does not scale as Ω^{-1} , as could naively be expected from Dicke's dimensional argument [32].

A. Example: FLRW space

Let us look at a very simple example, the spatially flat Friedmann-Lemaître-Robertson-Walker space with line element

$$d\tilde{s}^2 = -dt^2 + a^2(t) d\vec{x}^2. \quad (9)$$

By using the conformal time η related to the comoving time t by $dt = a d\eta$, the line element is manifestly conformal to the Minkowski one,

$$d\tilde{s}^2 = a^2(\eta) (-d\eta^2 + dr^2 + r^2 d\Omega_{(2)}^2) = \Omega^2 ds^2, \quad (10)$$

where we have used spherical coordinates $(\eta, r, \theta, \varphi)$ for the Minkowski line element ds^2 and the conformal factor $\Omega = a(\eta)$ preserves the spherical symmetry about every spatial point. A sphere of radius r (areal radius) in Minkowski space has vanishing Misner-Sharp-Hernandez/Hawking-Hayward mass, $M_{MSH} = 0$, while the mass of the corresponding sphere in FLRW space is, according to eq. (7),

$$\tilde{M}_{MSH} = a M_{MSH} - \frac{r^3}{2a} (-a_{,\eta}^2) - r^2 \nabla^c a(\eta) \nabla_c r = \frac{r^3}{2a} a_{,\eta}^2. \quad (11)$$

Since $a_{,\eta} = a da/dt \equiv a\dot{a}$, and the areal radius of FLRW space is $\tilde{R} = ar$, one obtains from eq. (11) that

$$\tilde{M}_{MSH} = \frac{4\pi}{3} \tilde{R}^3 \rho, \quad (12)$$

the well known expression of the Hawking-Hayward quasi-local mass in this space [15], corresponding to the mass of cosmic fluid enclosed in a sphere of proper radius \tilde{R} .

B. Example: the Sultana-Dyer cosmological black hole

The Sultana-Dyer spacetime [36] is a an inhomogeneous and time-dependent solution of GR interpreted as a black hole embedded in a spatially flat FLRW “background” (the quotation marks are compulsory since one cannot covariantly split a metric into a “background” and a “deviation” from it due to the non-linearity of the Einstein equations). The matter source is composed of two non-interacting perfect fluids, a timelike dust and a null dust [36]. The energy density becomes negative at late times near an event horizon. This solution has been studied as an example of a time-dependent black hole horizon for which the Hawking temperature can be derived explicitly [37] to shed light on the Hawking effect and the thermodynamics of time-evolving horizons. In this context, it is not important that the metric arises as a solution of GR with ill-behaved matter, or even that the metric is a solution of GR or another theory of gravity. The line element of the Sultana-Dyer metric is usually written as

$$ds^2 = a^2(\tau) \left[-d\tau^2 + dr^2 + r^2 d\Omega_{(2)}^2 + \frac{2m}{r} (d\tau + dr)^2 \right]. \quad (13)$$

What is interesting here is that the Sultana-Dyer metric can be obtained by conformally transforming the Schwarzschild solution. The line element can be rewritten as [36, 37]

$$d\tilde{s}^2 = a^2(\tau) \left[-\left(1 - \frac{2m}{r}\right) d\eta^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\Omega_{(2)}^2 \right], \quad (14)$$

where $a(\tau) = \tau^2$ is the scale factor of the spatially flat FLRW “background” (to which the solution reduces when $m \rightarrow 0$ or for $r \rightarrow +\infty$) and [36, 37]

$$\tau(\eta, r) = \eta + 2m \ln \left(\frac{r}{2m} - 1 \right). \quad (15)$$

The comoving time t of the FLRW “background” is given by $dt = a d\eta$, where η is the conformal time. This example contains the previous FLRW example as the trivial case $m = 0$. The FLRW “background” has scale factor $a(t) \sim t^{2/3}$ (the time evolution caused by a dust) in terms of comoving time. The Misner-Sharp-Hernandez

mass of a sphere of radius r is easily computed from the line element (14) using the definition (4) [37]:

$$\tilde{M}_{MSH} = ma - 2mra_{,\tau} + \frac{r^3 a_{,\tau}^2}{2a} \left(1 + \frac{2m}{r}\right). \quad (16)$$

Since the Sultana-Dyer metric is conformal to the Schwarzschild one which has $M_{MSH} = m$ and $R = r$ with $\Omega = a$, we can apply eq. (7) which gives, using eq. (15)

$$\begin{aligned} \tilde{M}_{MSH} &= ma - \frac{r^3}{2a} \left[g^{00} (a_{,\tau} \tau_{,\eta})^2 + g^{11} (a_{,\tau} \tau_{,r})^2 \right] \\ &\quad - r^2 g^{11} a_{,\tau} \tau_{,r} \\ &= ma + \frac{r^3 a_{,\tau}^2}{2a} \left(1 + \frac{2m}{r}\right) - 2mra_{,\tau}. \end{aligned} \quad (17)$$

Hence, also for this example, eq. (7) gives the correct result (16) computed independently in Ref. [37]. The Misner-Sharp-Hernandez mass does not scale as Ω^{-1} , rather it has a contribution which scales as Ω (comoving with the cosmic substratum) plus two contributions of opposite signs which depend on the expansion rate and on the expansion rate squared, respectively.

The interpretation of the result (17) (not attempted in Ref. [37]) is not straightforward. One can use the identity $a_{,\eta} = a_{,\tau} = a \partial a / \partial t \equiv a \dot{a}$, the definition of the (comoving) Hubble parameter $H \equiv \dot{a}/a$, and straightforward algebra to write

$$\tilde{M}_{MSH} = ma \left(1 - H\tilde{R}\right)^2 + \frac{H^2 \tilde{R}^3}{2}, \quad (18)$$

where $\tilde{R} = ar$ is the areal radius of the Sultana-Dyer spacetime. According to eq. (18), the mass \tilde{M}_{MSH} consists of two contributions. The first contribution is the mass $m = M_{MSH}$ of the Schwarzschild “seed” metric scaled by the conformal factor a but diluted by the expansion of the universe by the factor $(1 - H\tilde{R})^2$ (this

factor vanishes at $\tilde{R} = H^{-1}$, the radius that the cosmological horizon would have were the central inhomogeneity absent). It is not entirely clear how to interpret the competing effects of the factors a and $(1 - H\tilde{R})^2$

in this contribution to \tilde{M}_{MSH} . The second contribution to \tilde{M}_{MSH} can be written as $\frac{4\pi}{3} \tilde{R}^3 \rho$, where $\rho = \frac{3H^2}{8\pi}$ is the density that the cosmological fluid would have in the absence of the central inhomogeneity, to which the real density reduces for values of the radial coordinate $r \gg m$. This second contribution is interpreted as the mass of the cosmological fluid contained in the sphere of areal radius \tilde{R} (or coordinate radius r). In any case, the expression (18) makes it clear that the Misner-Sharp-Hernandez mass \tilde{M}_{MSH} of the Sultana-Dyer spacetime is always positive if the mass m of the seed Schwarzschild

metric is positive (or zero, which corresponds to g_{ab} being the Minkowski metric and \tilde{g}_{ab} the spatially flat FLRW space).

III. BEHAVIOR OF THE KODAMA VECTOR, SURFACE GRAVITY, AND TEMPERATURE UNDER CONFORMAL TRANSFORMATIONS

In the spacetime with spherically symmetric metric (1) the Kodama vector is given by

$$K^a = \frac{1}{\sqrt{AB}} \left(\frac{\partial}{\partial t} \right)^a, \quad (19)$$

with a similar expression holding in the rescaled spherical metric (2). In order to use the tilded version of this expression, the metric (2) must be recast in terms of the areal radius \tilde{R} in the form

$$d\tilde{s}^2 = -\tilde{A}d\tilde{T}^2 + \tilde{B}d\tilde{R}^2 + \tilde{R}^2 d\Omega_{(2)}^2, \quad (20)$$

where \tilde{T} will be, in general, a new time coordinate which is a function of both t and R . Since $\Omega = \Omega(t, R)$ to preserve the spherical symmetry and $\tilde{R} = \Omega R$, the relation between differentials

$$dR = \frac{d\tilde{R} - \Omega_{,t} R dt}{\Omega_{,R} R + \Omega} \quad (21)$$

follows. Inserting this relation in the line element (2) yields

$$\begin{aligned} d\tilde{s}^2 &= - \left[\Omega^2 A - \frac{\Omega_{,t}^2 R^2 \Omega^2 B}{(\Omega_{,R} R + \Omega)^2} \right] dt^2 \\ &\quad + \frac{\Omega^2 B}{(\Omega_{,R} R + \Omega)^2} d\tilde{R}^2 \\ &\quad - \frac{2\Omega^2 \Omega_{,t} B R}{(\Omega_{,R} R + \Omega)^2} dt d\tilde{R} + \tilde{R}^2 d\Omega_{(2)}^2. \end{aligned} \quad (22)$$

In order to achieve the diagonal form of the line element, the $dt d\tilde{R}$ cross-term must be eliminated by introducing a new time coordinate $T(t, R)$ defined by

$$dT = \frac{1}{F} \left(dt + \beta d\tilde{R} \right), \quad (23)$$

where $\beta(t, \tilde{R})$ is a function to be determined and $F(t, \tilde{R})$ is an integrating factor which must satisfy the equation

$$\frac{\partial}{\partial \tilde{R}} \left(\frac{1}{F} \right) = \frac{\partial}{\partial t} \left(\frac{\beta}{F} \right) \quad (24)$$

to guarantee that dT is an exact differential. The substitution of $dt = F dT - \beta d\tilde{R}$ into eq. (22) yields

$$d\tilde{s}^2 = - \left[\Omega^2 A - \frac{\Omega_{,t}^2 R^2 \Omega^2 B}{(\Omega_{,R} R + \Omega)^2} \right] F^2 dT^2 + \left\{ -\beta^2 \left[\Omega^2 A - \frac{\Omega_{,t}^2 R^2 \Omega^2 B}{(\Omega_{,R} R + \Omega)^2} \right] + \frac{\Omega^2 B}{(\Omega_{,R} R + \Omega)^2} \right. \\ \left. + \frac{2\beta\Omega_{,t}\Omega^2 BR}{(\Omega_{,R} R + \Omega)^2} \right\} d\tilde{R}^2 + 2F \left\{ \beta \left[\Omega^2 A - \frac{\Omega_{,t}^2 R^2 \Omega^2 B}{(\Omega_{,R} R + \Omega)^2} \right] - \frac{\Omega_{,t}\Omega^2 RB}{(\Omega_{,R} R + \Omega)^2} \right\} dT d\tilde{R} + \tilde{R}^2 d\Omega_2^2. \quad (25)$$

The choice

$$\beta(t, R) = \frac{\Omega_{,t}\Omega^2 BR}{\left[\Omega^2 A - \frac{\Omega_{,t}^2 R^2 \Omega^2 B}{(\Omega_{,R} R + \Omega)^2} \right] (\Omega_{,R} R + \Omega)^2} \quad (26)$$

then turns the line element into the diagonal form

$$d\tilde{s}^2 = - \left[\Omega^2 A - \frac{\Omega_{,t}^2 \Omega^2 BR^2}{(\Omega_{,R} R + \Omega)^2} \right] F^2 dT^2 + \frac{\Omega^2 B}{(\Omega_{,R} R + \Omega)^2} \left\{ 1 + \frac{\Omega_{,t}^2 \Omega^2 BR^2}{(\Omega_{,R} R + \Omega)^2 \left[\Omega^2 A - \frac{\Omega_{,t}^2 \Omega^2 BR^2}{(\Omega_{,R} R + \Omega)^2} \right]} \right\} d\tilde{R}^2 + \tilde{R}^2 d\Omega_2^2. \quad (27)$$

The comparison of this line element with eq. (20) yields

$$\tilde{A} = \left[\Omega^2 A - \frac{\Omega_{,t}^2 \Omega^2 BR^2}{(\Omega_{,R} R + \Omega)^2} \right] F^2, \quad (28)$$

$$\tilde{B} = \frac{B\Omega^2}{(\Omega_{,R} R + \Omega)^2} \left\{ 1 + \frac{\Omega_{,t}^2 \Omega^2 BR^2}{(\Omega_{,R} R + \Omega)^2 \left[\Omega^2 A - \frac{\Omega_{,t}^2 \Omega^2 BR^2}{(\Omega_{,R} R + \Omega)^2} \right]} \right\}. \quad (29)$$

In the non-tilde world (1) the unit vector in the time direction is $u^a \equiv \left(\frac{\partial}{\partial t}\right)^a$ and has components $u^\mu = \left(\frac{1}{\sqrt{A}}, \vec{0}\right)$, while the corresponding vector $v^a \equiv \left(\frac{\partial}{\partial T}\right)^a$ in the tilde world (2) has components $v^\mu = \left(\frac{1}{\sqrt{\tilde{A}}}, \vec{0}\right)$. The corresponding Kodama vectors have components $K^\mu = \left(\frac{1}{A\sqrt{B}}, \vec{0}\right)$ (in coordinates (t, R, θ, φ)) and $\tilde{K}^\mu = \left(\frac{1}{\tilde{A}\sqrt{\tilde{B}}}, \vec{0}\right)$ (in coordinates $(T, \tilde{R}, \theta, \varphi)$). Therefore, the only non-vanishing component of the Kodama vector satisfies

$$\tilde{K}^0 = \frac{A(\Omega_{,R} R + \Omega)}{\tilde{A}\Omega \sqrt{1 + \frac{\Omega_{,t}^2 \Omega^2 BR^2}{(\Omega_{,R} R + \Omega)^2 + \left[\Omega^2 A - \frac{\Omega_{,t}^2 \Omega^2 BR^2}{(\Omega_{,R} R + \Omega)^2} \right]}}} K^0. \quad (30)$$

The norm squared of the Kodama vector in the non-tilde metric is

$$K^a K_a = -A(K^0)^2 \quad (31)$$

and the norm squared of \tilde{K}^a is

$$\tilde{K}^a \tilde{K}_a = -\frac{A(\Omega_{,R} R + \Omega)^2 (K^0)^2}{\Omega^4 F^2}$$

$$= \frac{(\Omega_{,R} R + \Omega)^2}{\Omega^4 F^2} (K^a K_a). \quad (32)$$

The factor $\frac{(\Omega_{,R} R + \Omega)^2}{\Omega^4 F^2}$ is non-negative and therefore \tilde{K}^a has the same causal nature as K^a except possibly where this factor vanishes, which is excluded by the following considerations. It would seem that eq. (32) singles out a class of special conformal transformations, those which satisfy identically the equation

$$\Omega_{,R} R + \Omega = 0, \quad (33)$$

but these transformations are physically irrelevant. In fact, eq. (33) is immediately integrated to

$$\Omega(t, R) = \frac{f(t)}{R} \quad (34)$$

with $f(t)$ a positive function of the time coordinate. In addition to being ill-defined at $R = 0$ and $R = +\infty$, this transformation changes the causal character of the areal radius since eq. (3) then gives the areal radius $\tilde{R} = f(t)$ and \tilde{R} becomes a timelike coordinate. We will not consider these conformal transformations further and we will assume that $\Omega_{,R} R + \Omega$ does not vanish identically.

The Kodama surface gravity κ_K is given as follows: let the spherically symmetric spacetime metric be given by

$$ds^2 = h_{ij} dx^i dx^j + R^2 d\Omega_2^2, \quad (35)$$

where h_{ij} ($i, j = 0, 1$) is the 2-metric in the space orthogonal to the 2-spheres of symmetry and R is the areal radius; then

$$\kappa_K = \frac{1}{2} \square_{(h)} R = \frac{1}{2\sqrt{-h}} \partial_i \left(\sqrt{-h} h^{ij} \partial_j R \right) \quad (36)$$

where h is the determinant of h_{ij} . Since, in our notations, it is $h_{ij} = \text{diag}(-A, B)$, $h = -AB$, $\tilde{h}_{ij} = \text{diag}(-\Omega^2 A, \Omega^2 B)$, $\tilde{h} = \Omega^4 h$, and $\tilde{R} = \Omega R$, the Kodama surface gravity in the tilded world is

$$\begin{aligned} \tilde{\kappa}_K &= \frac{1}{2\sqrt{-\tilde{h}}} \partial_i \left(\sqrt{-\tilde{h}} \tilde{h}^{ij} \partial_j \tilde{R} \right) \\ &= \frac{1}{2\Omega^2 \sqrt{-h}} \partial_i \left(\sqrt{-h} h^{ij} \Omega \partial_j R + \sqrt{-h} h^{ij} R \partial_j \Omega \right) \\ &= \frac{\kappa_K}{\Omega} + \frac{1}{2\Omega^2 \sqrt{-h}} \partial_i \left(\sqrt{-h} h^{ij} R \partial_j \Omega \right) \\ &\quad + \frac{1}{2\Omega^2} h^{ij} \partial_i \Omega \partial_j R. \end{aligned} \quad (37)$$

To the best of our knowledge, this transformation formula was first presented in the static case in Ref. [28] (see their eq. (29) and note that the last term in eq. (37) vanishes on the apparent horizon if $\Omega = \Omega(R)$ only). We can further write it as

$$\begin{aligned} \tilde{\kappa}_K &= \frac{\kappa_K}{\Omega} \\ &\quad + \frac{1}{2\Omega^2 \sqrt{AB}} \left\{ -R \left[\Omega_{,tt} \sqrt{\frac{B}{A}} + \frac{\Omega_{,t} (AB_{,t} - BA_{,t})}{2A\sqrt{AB}} \right] \right. \\ &\quad + \sqrt{\frac{A}{B}} \Omega_{,R} + R \sqrt{\frac{A}{B}} \Omega_{,RR} \\ &\quad \left. + \Omega_{,RR} \frac{(BA_{,R} - AB_{,R})}{B\sqrt{AB}} \right\} + \frac{\Omega_{,R}}{2\Omega^2 B}. \end{aligned} \quad (38)$$

Therefore, the Kodama temperature

$$T_K = \frac{\kappa_K}{2\pi} \quad (39)$$

(evaluated at the apparent horizon) does not scale simply as $\tilde{T}_K \sim T_K/\Omega$, as it could be expected naively from Dicke's dimensional argument ([32], see also [38]). However, a simplification is possible in the case of cosmological black holes obtained via a conformal transformation of a stationary black hole with conformal factor equal to the scale factor a of a "background" FLRW space. In this case the conformal factor depends only on time and the transformation property of the Kodama temperature simplifies to

$$\tilde{T}_K = \frac{T_K}{\Omega} - \frac{R}{4\pi\Omega^2} \left[\frac{\Omega_{,tt}}{A} + \frac{\Omega_{,t} (AB_{,t} - BA_{,t})}{2A^2 B} \right]. \quad (40)$$

It seems intuitive that an apparent horizon evolving arbitrarily fast would not constitute a system in thermodynamical equilibrium and non-equilibrium thermodynamics would be necessary, as opposed to a stationary black hole which is instead a system in thermodynamical equilibrium. These would be bad news for the thermodynamics of apparent/trapping horizons; however, it is reasonable to expect that an adiabatic approximation in which the spacetime and its apparent horizons evolve slowly constitutes a small deviation from thermal equilibrium amenable to a simplified description. In this case, eq. (40) reads

$$\tilde{T}_K = \frac{T_K}{\Omega} + \dots \quad (41)$$

where the ellipsis denote small corrections which can be controlled in the above-mentioned adiabatic approximation. The need for such an adiabatic approximation to make sense of Hawking radiation and the thermodynamics of time-evolving horizons has been stressed clearly as a requisite for the applicability of the Hamilton-Jacobi method in the tunneling approach [39] to the computation of T [28] and in the renormalization of the scalar field stress-energy tensor in the background of cosmological black holes [37].

A. Locating the tilded apparent horizons

The apparent horizons of a spherically symmetric spacetime (1) are given by the roots of the equation $\nabla^c R \nabla_c R = 0$, where R is its areal radius, which is equivalent to $g^{RR} = 0$ (e.g., [22]). In the tilded world (2) the tilded apparent horizons are given by $\tilde{g}^{\tilde{R}\tilde{R}} = 0$ or, according to eq. (29)

$$\frac{A(\Omega_{,R}R + \Omega)^2 - \Omega_{,t}^2 B R^2}{\Omega^2 AB} = 0 \quad (42)$$

if $\Omega_{,R}R + \Omega \neq 0$. The possible solutions of this equation include the solutions of

$$\frac{1}{B} = 0 \quad (43)$$

(corresponding to $g^{RR} = 0$ in the non-tilded spacetime (1)); those of

$$A(\Omega_{,R}R + \Omega)^2 = \Omega_{,t}^2 B R^2; \quad (44)$$

and $A = \infty$. Let us now look at some examples to make sense of the formulas derived so far.

B. Example: FLRW space

Let us consider again the spatially flat FLRW space conformal to Minkowski space as an example. We have

here $A = B = 1$, $F = 1$, $\Omega = a(\eta)$, $R = r$, $\tilde{R} = ar$, $K^a = (\partial/\partial\eta)^a$, and

$$\tilde{K}^0 = \frac{aK^0}{a\tilde{A}\sqrt{1 + \frac{a_\eta a^2 r^2}{a^2 \left[a^2 - \frac{a_\eta^2 a^2 r^2}{a^2} \right]}}}. \quad (45)$$

The relation $a_{,\eta} = a\dot{a}$ with $H \equiv \dot{a}/a$ gives

$$\tilde{K}^\mu = \left(\frac{K^0}{a^2 \sqrt{1 - H^2 \tilde{R}^2}}, \vec{0} \right) \quad (46)$$

and the norm squared

$$\tilde{K}^a \tilde{K}_a = -\frac{1}{a^2}. \quad (47)$$

The Kodama vector of FLRW space is timelike ($\tilde{K}^a \tilde{K}_a < 0$) in the region below the cosmological horizon $\tilde{R} < 1/H$ in which it is defined.

C. Example: the Sultana-Dyer spacetime

Let us return to the example of the Sultana-Dyer black hole and let us locate the apparent horizons using eqs. (43)-(44). For comparison, we refer to the study of the causal structure of this spacetime in Ref. [37]. The apparent horizons are the roots of the equation $1 - \frac{2\tilde{M}_{MSH}}{\tilde{R}} = \tilde{g}^{ab} \tilde{\nabla}_a \tilde{R} \tilde{\nabla}_b \tilde{R} = \tilde{g}^{\tilde{R}\tilde{R}} = 0$. Since $\tilde{R} = ar$, the expression (16) of the Misner-Sharp-Hernandez mass \tilde{M}_{MSH} in the Sultana-Dyer metric gives

$$2ma + \frac{r^3 a_{,\tau}^2}{a} \left(1 + \frac{2m}{r} \right) - 4mra_{,\tau} = ar. \quad (48)$$

Using the fact that $a(\tau) = \tau^2$ and $a_{,\tau} = 2\tau$, one obtains

$$1 - \frac{2m}{r} = \frac{4r^2}{\tau^2} \left(1 + \frac{2m}{r} \right) - \frac{8m}{\tau} \quad (49)$$

as the equation locating the apparent horizons (which coincides with eq. (3.10) of [37]). This is the cubic equation for r

$$4r^3 + 8mr^2 - (8m + \tau)\tau r + 2m\tau^2 = 0, \quad (50)$$

the real positive roots of which are [37]

$$r_1 = \frac{-4m - \tau + \sqrt{\tau^2 + 24m\tau + 16m^2}}{4}, \quad (51)$$

$$r_2 = \frac{\tau(\eta, r)}{2}, \quad (52)$$

with $r_1 < r_2$. In addition, the surface $r = 2m$, the null event horizon of the Schwarzschild seed metric, remains an event horizon for the Sultana-Dyer metric (see Ref. [37] for a conformal diagram of this spacetime).

Let us proceed now using eqs. (43)-(44). Eq. (43) gives again $r_{EH} = 2m$, corresponding to areal radius $\tilde{R}_{EH} = ar_{EH} = 2m\tau^2$ and to an event horizon which is exactly comoving with the cosmic substratum (this fact, not noted before, has some relevance for the long-standing debate of the effect of the cosmological expansion on local systems, see Ref. [40] for a recent review). Eq. (44) gives

$$\left(1 - \frac{2m}{r} \right) (a_\tau \tau_{,r} r + a)^2 = \frac{(a_{,\tau} \tau_{,t})^2 r^2}{1 - \frac{2m}{r}} \quad (53)$$

and

$$\left(1 - \frac{2m}{r} \right) \left(\frac{2ma_{,\tau}}{1 - \frac{2m}{r}} + a \right) = \pm a_{,\tau} r. \quad (54)$$

The upper (positive) sign applies if $r > 2m$. Strictly speaking, the region $r \leq 2m$ is not allowed because we are using here the conformal transformation (14) with $a = \tau^2$ and τ given by eq. (15), which requires $r > 2m$. However, one can consider the continuation of the metric (14) to the region $r \leq 2m$, hence we keep also the lower (negative) sign in eq. (54) for the moment. Then we obtain

$$\epsilon a_{,\tau} r^2 + (2ma_{,\tau} + a)r - 2ma = 0 \quad (55)$$

and

$$r = \frac{-(4m + \tau) \pm \sqrt{(4m + \tau)^2 + 16m\tau}}{4\epsilon}, \quad (56)$$

where $\epsilon = \pm 1$ keeps track of both signs coming from eq. (54). For $\epsilon = -1$ we obtain the roots

$$r_1 = \frac{\tau}{2}, \quad r_2 = 2m, \quad (57)$$

while for $\epsilon = +1$ one obtains

$$r_3 = - \left[\frac{4m + \tau + \sqrt{\tau^2 + 24m\tau + 16m^2}}{4} \right], \quad (58)$$

which is negative and is discarded, and

$$r_4 = \frac{-(4m + \tau) + \sqrt{\tau^2 + 24m\tau + 16m^2}}{4}, \quad (59)$$

which is the radius of the black hole apparent horizon found by Saida, Harada, and Maeda (eq. (3.11a) of [37]). They do not use the form of the Sultana-Dyer metric explicitly conformal to Schwarzschild and they do not need to invoke any continuation, although they arrive to the cubic equation (50) for the apparent horizon radii which is less straightforward to solve than the quadratic equation (55). In terms of the areal radius $R = ar$, the apparent horizon radii are

$$R_1 = \frac{\tau^3}{2}, \quad R_2 = R_{EH} = 2m\tau^2, \quad (60)$$

and

$$R_4 = \frac{-4m - \tau + \sqrt{\tau^2 + 24m\tau + 16m^2}}{4} \tau^2. \quad (61)$$

Ultimately these are implicit equations for the radii of the apparent horizons in terms of tilded quantities t and \tilde{R} .

The Hawking temperature of the Sultana-Dyer black hole was studied in Ref. [37] by computing the renormalized stress-energy tensor of a free, conformally coupled, scalar field in this spacetime. The result is the effective temperature

$$\tilde{T} = \frac{1}{8\pi m a} + \dots \quad (62)$$

where the ellipsis denotes corrections which are small in the limit of a slowly evolving black hole [37]. Since $T = \frac{1}{8\pi m}$ is the Hawking temperature of the “seed” Schwarzschild black hole, one can write

$$\tilde{T} = \frac{T}{\Omega} + \dots; \quad (63)$$

The conformal factor of the Sultana-Dyer black hole does not depend on the radial coordinate and, in the adiabatic approximation in which its time variation is small, the Hawking temperature does have the scaling behaviour expected on dimensional grounds. This scaling law, however, will break down as soon as the conformal transformation is allowed to be radial-dependent, or the apparent horizon is allowed to vary rapidly.

D. Asymptotically flat spacetimes

The transformation properties of the Hawking temperature of black hole horizons under conformal transformations have been the subject of much literature, in which asymptotically flat spacetimes were considered together with conformal transformations which reduce to the identity ($\Omega \rightarrow 1$) at spatial infinity $R \rightarrow +\infty$ [18, 20]. This last property follows from the necessity of normalizing to unity the Killing vector at spatial infinity, where observers detecting the Hawking flux are located. It is clear that this case is not relevant when the interest is on cosmological black holes, however it has its importance and it is worth at least a brief mention here. In this special case the transformation property (7) of the Misner-Sharp-Hernandez mass gives $\tilde{M}_{MSH} = M_{MSH}$ at infinity, *i.e.*, this mass notion is a conformal invariant when evaluated at large spatial distances from the asymptotically flat inhomogeneity.

Similarly, eq. (24) relating the Kodama vectors in the tilded and non-tilded worlds reduces, in this special case, to $\tilde{K}^a \simeq K^a$ at infinity, or wherever one wants to normalize the timelike Killing vector to unity. This fact is important because the Kodama vector defines a specific (“Kodama”) surface gravity and a Kodama temperature

which is regarded by many authors as the physical temperature of a black hole. The invariance of the Kodama vector under restricted conformal transformations therefore implies the conformal invariance of the temperature.

IV. KERR-SCHILD TRANSFORMATIONS

Another technique employed to generate cosmological black holes embedded in FLRW “backgrounds” is that of (generalized) Kerr-Schild transformations [41]: beginning from the “seed” metric g_{ab} , this transformation generates

$$g_{ab} \rightarrow \bar{g}_{ab} = g_{ab} + 2\lambda l_a l_b, \quad (64)$$

where λ is a function of the spacetime point and l^a is a null and geodesic vector of the metric g_{ab} ,

$$g_{ab} l^a l^b = 0, \quad l^a \nabla_a l^b = 0. \quad (65)$$

It follows that l^a is null and geodesic also with respect to the “new” metric \bar{g}_{ab} ,

$$\bar{g}_{ab} l^a l^b = g_{ab} l^a l^b + 2\lambda l_a l^a l_b l^b = 0, \quad (66)$$

$$\bar{g}_{ab} l^a \nabla^b l^c = 0. \quad (67)$$

The tensor

$$\bar{g}^{ab} = g^{ab} - 2\lambda l^a l^b \quad (68)$$

is the inverse of the metric \bar{g}_{ab} , as follows from the fact that

$$\begin{aligned} \bar{g}^{ab} \bar{g}_{bc} &= (g^{ab} - 2\lambda l^a l^b) (g_{bc} + 2\lambda l_b l_c) \\ &= g^{ab} g_{bc} + 2\lambda g^{ab} l_b l_c - 2\lambda g_{bc} l^a l^b - 4\lambda^2 l^a l^b l_b l_c \\ &= \delta_c^a. \end{aligned} \quad (69)$$

The *caveat* raised for metrics generated by conformal transformations applies here: it is not trivial that a seed metric which solves the Einstein equations will generate another solution of the same equations corresponding to a “reasonable” form of mass-energy, and most of the cosmological black hole solutions generated by this technique in the literature have unphysical stress-energy tensors at least in some regions of the spacetime manifold [41]. However, this technique is not to be discarded, for example it generates the Schwarzschild and Reissner-Nordström solutions using Minkowski space as the seed.

We can ask ourselves how the Misner-Sharp-Hernandez mass transforms under Kerr-Schild transformations which use a spherically symmetric metric of the form (1) and respects the spherical symmetry, *i.e.*, $\lambda = \lambda(t, R)$ and $l^a = l^a(t, R)$ in these coordinates. We have

$$\begin{aligned} d\bar{s}^2 &= ds^2 + 2\lambda l_a l_b dx^a dx^b = -[A - 2\lambda(l_0)^2] dt^2 \\ &\quad + [B + 2\lambda(l_1)^2] dR^2 + 4\lambda l_0 l_1 dt dR + R^2 d\Omega_{(2)}^2, \end{aligned} \quad (70)$$

where the “new” areal radius coincides with the “old” one, $\bar{R} = R$. In order to eliminate the $dt dR$ cross-term we introduce again a new time coordinate T defined by $dT = \frac{1}{F}(dt + \beta dR)$, where $\beta(t, R)$ is a function to be determined and $F(t, R)$ is an integrating factor satisfying eq. (24). The relation $dt = F dT - \beta dR$ gives

$$\begin{aligned} d\bar{s}^2 = & - [A - 2\lambda(l_0)^2] F^2 dT^2 \\ & + \{B + 2\lambda(l_1)^2 - \beta^2 [A - 2\lambda(l_0)^2] - 4\lambda l_0 l_1 \beta\} dR^2 \\ & + 2F \{ \beta [A - 2\lambda(l_0)^2] + 2\lambda l_0 l_1 \} dT dR + R^2 d\Omega_{(2)}^2. \end{aligned}$$

By imposing that

$$\beta(t, R) = \frac{-2\lambda l_0 l_1}{A - 2\lambda(l_0)^2} \quad (71)$$

the metric is diagonalized and assumes the form

$$\begin{aligned} d\bar{s}^2 = & - [A - 2\lambda(l_0)^2] F^2 dT^2 \\ & + \left\{ B + 2\lambda(l_1)^2 + \frac{4\lambda^2(l_0)^2(l_1)^2}{A - 2\lambda(l_0)^2} \right\} dR^2 + R^2 d\Omega_{(2)}^2. \end{aligned} \quad (72)$$

Given that $\bar{R} = R$, the Misner-Sharp-Hernandez mass of the barred spacetime is defined by

$$1 - \frac{2\bar{M}_{MSH}}{R} = \bar{g}^{ab} \nabla_a R \nabla_b R = \bar{g}^{RR} \quad (73)$$

which gives, using eq. (68),

$$\bar{M}_{MSH} = M_{MSH} + \lambda(l^1)^2 R. \quad (74)$$

The “new” Misner-Sharp-Hernandez mass of a sphere of radius R is always non-negative if M_{MSH} is non-negative.

It is straightforward to locate the apparent horizons of the spherical metric \bar{g}_{ab} in terms of those of the metric g_{ab} . These apparent horizons are the roots of $\bar{g}^{ab} \bar{\nabla}_a R \bar{\nabla}_b R = 0$, or $\bar{g}^{RR} = 0$, which gives

$$g^{RR} - 2\lambda(l^1)^2 = 0. \quad (75)$$

In practice, a null future-oriented vector is defined up to a positive constant and one can set $l^1 = 1$ in eqs. (74) and (75) without loss of generality. It is clear then that it is the function λ which determines the deviation of the barred quantities from the non-barred ones.

A. Behavior of the Kodama temperature under Kerr-Schild transformations

The behavior of the Kodama temperature under Kerr-Schild transformations preserving the spherical symmetry of the metric is derived using eq. (36) for the Kodama surface gravity. By using the fact that $\bar{h}_{ij} = h_{ij} + 2\lambda l_i l_j$ and $\bar{h}^{ij} = h^{ij} - 2\lambda l^i l^j$, one finds

$$\begin{aligned} \bar{h} &= [-A + 2\lambda(l_0)^2] [B + 2\lambda(l_1)^2] - 4\lambda^2(l_0)^2(l_1)^2 \\ &= -AB + 2\lambda [-A(l_1)^2 + B(l_0)^2] \\ &= h \left\{ 1 - 2\lambda \left[-\frac{(l_1)^2}{B} + \frac{(l_0)^2}{A} \right] \right\}, \end{aligned} \quad (76)$$

where $h = -AB$. Remembering that l^a is a null vector of the metric g_{ab} , one obtains $-A(l^0)^2 + B(l^1)^2 = 0$ and, using $l_0 = g_{00}l^0 = -Al^0$, $l_1 = g_{11}l^1 = Bl^1$, it is $-\frac{(l_0)^2}{A} + \frac{(l_1)^2}{B} = 0$ and the exact relation

$$\bar{h} = h \quad (77)$$

follows. Eq. (36) now yields

$$\begin{aligned} \bar{\kappa}_K &= \frac{1}{2\sqrt{-\bar{h}}} \left\{ \partial_t \left[\sqrt{-\bar{h}} (\bar{h}^{00} \partial_t R + \bar{h}^{01} \partial_R R) \right] + \partial_R \left[\sqrt{-\bar{h}} (\bar{h}^{11} \partial_R R + \bar{h}^{10} \partial_t R) \right] \right\} \\ &= \frac{1}{2\sqrt{-\bar{h}}} \partial_R \left(\frac{\sqrt{-\bar{h}}}{B} \right) - \frac{1}{\sqrt{AB}} \left[\partial_t \left(\lambda l^0 l^1 \sqrt{AB} \right) + \partial_R \left(\lambda l^1 l^1 \sqrt{AB} \right) \right], \end{aligned} \quad (78)$$

or

$$\bar{\kappa}_K = \kappa_K - \frac{1}{\sqrt{AB}} \left[\partial_t \left(\lambda l^0 l^1 \sqrt{AB} \right) + \partial_R \left(\lambda l^1 l^1 \sqrt{AB} \right) \right]. \quad (79)$$

Again, one can set $l^1 = 1$ in the previous expressions.

B. Example: the Reissner-Nordström black hole

The Reissner-Nordström black hole with line element

$$d\bar{s}^2 = - \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right) dT^2 + \frac{dr^2}{1 - \frac{2m}{r} + \frac{Q^2}{r^2}} + r^2 d\Omega_{(2)}^2, \quad (80)$$

where m and Q are the mass and charge parameters, respectively, can be obtained by a Kerr-Schild transfor-

mation of the Minkowski space metric $ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{(2)}^2$ in spherical coordinates with

$$\lambda(t, r) = \frac{m}{r} - \frac{Q^2}{2r^2}, \quad (81)$$

$$l^\mu = (-1, 1, 0, 0) \quad (82)$$

in these coordinates. In fact, this transformation gives

$$d\bar{s}^2 = -\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 + \frac{2m}{r} - \frac{Q^2}{r^2}\right) dr^2 + r^2 d\Omega_{(2)}^2 + 2\left(\frac{2m}{r} - \frac{Q^2}{r^2}\right) dt dr. \quad (83)$$

This line element can be brought to the diagonal form (80) by performing the usual transformation to a new time coordinate T defined by $dT = \frac{1}{F}(dt + \beta dr)$, which yields

$$\begin{aligned} d\bar{s}^2 = & -\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) F^2 dT^2 \\ & + \left\{ -\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) \beta^2 + 1 + \frac{2m}{r} - \frac{Q^2}{r^2} \right. \\ & \left. - 2\left(\frac{2m}{r} - \frac{Q^2}{r^2}\right) \beta \right\} dr^2 \\ & + 2\left\{ \beta \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) + \left(\frac{2m}{r} - \frac{Q^2}{r^2}\right) \right\} F dT dr \\ & + r^2 d\Omega_{(2)}^2. \end{aligned} \quad (84)$$

The choice of the function

$$\beta(t, r) = \frac{-\left(\frac{2m}{r} - \frac{Q^2}{r^2}\right)}{1 - \frac{2m}{r} + \frac{Q^2}{r^2}} \quad (85)$$

brings the line element into the form

$$\begin{aligned} d\bar{s}^2 = & -\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) F^2 dT^2 \\ & + \frac{dr^2}{1 - \frac{2m}{r} + \frac{Q^2}{r^2}} + r^2 d\Omega_{(2)}^2. \end{aligned} \quad (86)$$

The integrating factor $F(t, r)$ must satisfy eq. (24), which for the static case under consideration reduces to

$$\frac{\partial}{\partial r} \left(\frac{1}{F} \right) = 0 \quad (87)$$

and is satisfied by the choice $F \equiv 1$, which finally brings the metric (86) into the familiar Reissner-Nordström form (80).

Our formula (74) for the Kerr-Schild transformation of the Misner-Sharp-Hernandez mass gives, using eqs. (81), (82), and $M_{MSH} = 0$, the mass of a sphere of radius r as

$$\bar{M}_{MSH} = m - \frac{Q^2}{2r}. \quad (88)$$

This is the well known expression of the Misner-Sharp-Hernandez mass of a sphere of radius r in the Reissner-Nordström spacetime, which can be obtained immediately from the definition $1 - \frac{2\bar{M}_{MSH}}{R} = \bar{g}^{ab} \bar{\nabla}_a R \bar{\nabla}_b R$.

The apparent horizons of the metric \bar{g}_{ab} can be obtained from those of the seed Minkowski metric (which has no apparent horizons) by using eq. (75) which yields, using eqs. (81) and (82),

$$g^{RR} - 2\lambda(l^1)^2 = 1 - \left(\frac{2m}{r} - \frac{Q^2}{r^2}\right) = 0 \quad (89)$$

or $r^2 - 2mr + Q^2 = 0$, which has the well known roots

$$r_{\pm} = m \pm \sqrt{m^2 - Q^2} \quad (90)$$

corresponding to the outer (event) horizon and to the inner (Cauchy) horizon, respectively.

Finally, let us check that the Reissner-Nordström surface gravity coincides with that given by eq. (79). The latter equation gives, using $\kappa_K = 0$ for Minkowski space, $AB = 1$ and $\lambda = \frac{m}{r} - \frac{Q^2}{2r^2}$,

$$\begin{aligned} \bar{\kappa}_K &= 0 - \left\{ \partial_t \left[\left(\frac{m}{r} - \frac{Q^2}{2r^2} \right) \right] + \partial_r \left[\left(\frac{m}{r} - \frac{Q^2}{2r^2} \right) \right] \right\} \\ &= \frac{m^2}{r^2} - \frac{Q^2}{r^3}. \end{aligned} \quad (91)$$

Direct use of the expression (36) gives

$$\begin{aligned} \bar{\kappa}_K &= \frac{1}{2} [\partial_t \bar{h}^{00} + \partial_r \bar{h}^{11}] = \frac{1}{2} \partial_r \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right) \\ &= \frac{m^2}{r^2} - \frac{Q^2}{r^3}, \end{aligned} \quad (92)$$

in agreement with the previous expression (91).

V. DISCUSSION AND CONCLUSIONS

The textbook concept of event horizon is suitable for stationary black holes and is central in the construction of black hole mechanics and thermodynamics but, due to its teleological nature, becomes useless for dynamical black holes where its best replacement currently available is the notion of apparent or trapping horizon. Here we restricted ourselves to spherical symmetry, for which the shortcoming of dependence of the apparent horizons on the spacetime foliation is less severe.

Conformal transformation techniques and Kerr-Schild transformations are commonly encountered in relation with dynamical black holes, in both General Relativity and alternative theories of gravity. When trying to extend the known theory of thermodynamics to apparent or trapping horizons in time-dependent situations [8], it is useful to know how various quantities (Misner-Sharp-Hernandez mass or quasi local Hawking-Hayward energy,

Kodama vector, surface gravity and temperature) transform under these transformations. We have addressed this question here: these quantities do not transform as one would expect on the basis of naive dimensional arguments [32], which is natural since, for example, the quasi-local mass of a sphere depends on both the matter in it and the gravitational energy, while the naive arguments are designed for test particles. We have provided examples to test the transformation formulas that we derived in more intuitive situations.

In spite of rather strong claims, it is not yet clear whether marginally trapped surfaces and apparent/trapping horizons are the best concepts to apply in the characterization of black holes but, *de facto*, they are used routinely in numerical relativity with the practical goal of producing templates for immediate use in

the detection of gravitational waves by giant laser interferometers such as *LIGO* and *VIRGO*. It is hoped that theoretical efforts be multiplied to understand better key concepts such as the practical definition of black hole, horizons, their theoretical thermodynamics, and the limitations of the theoretical constructs used.

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